Negative Inverse Temperatures in 2D Fluid Mechanics

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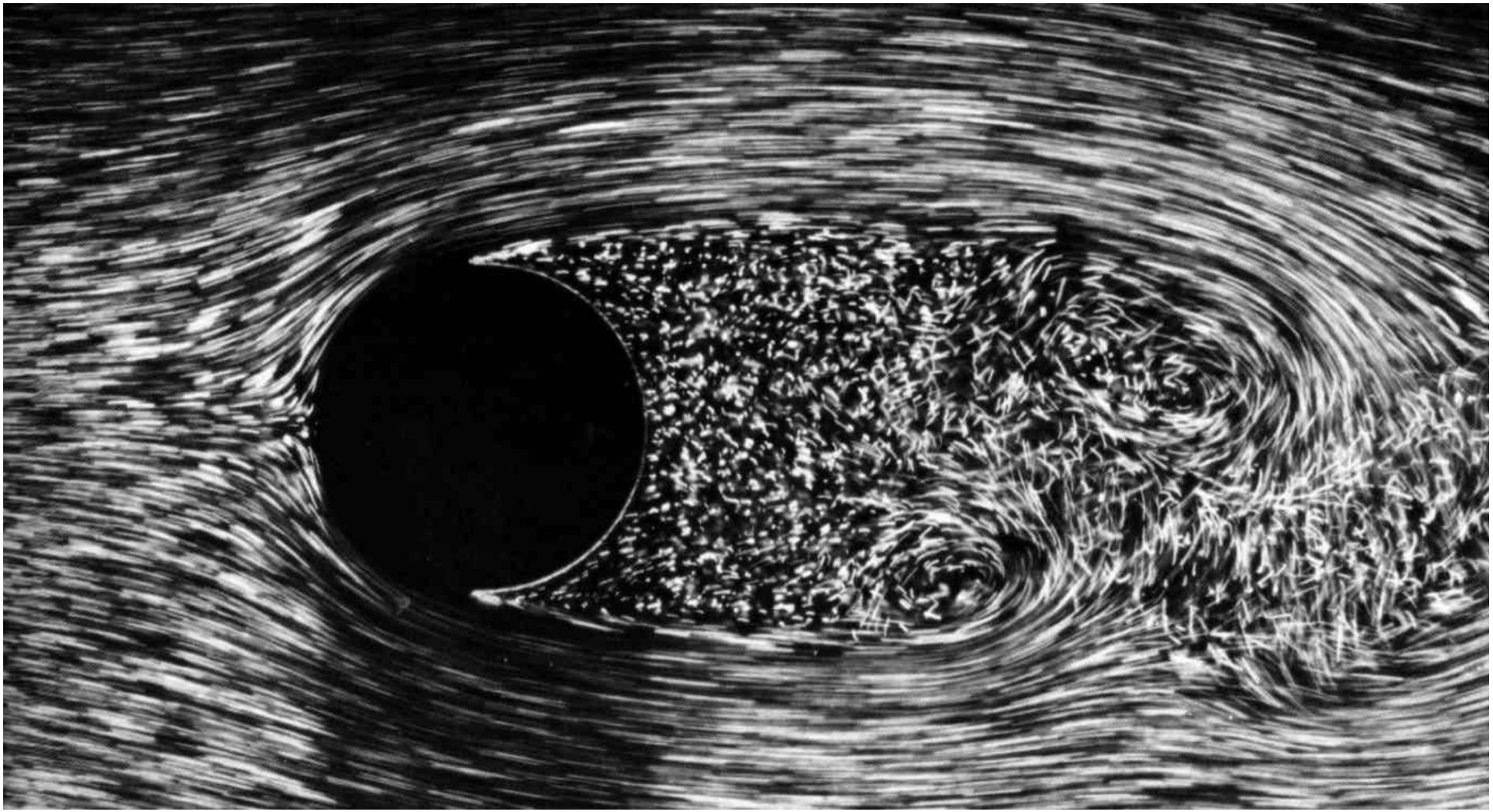
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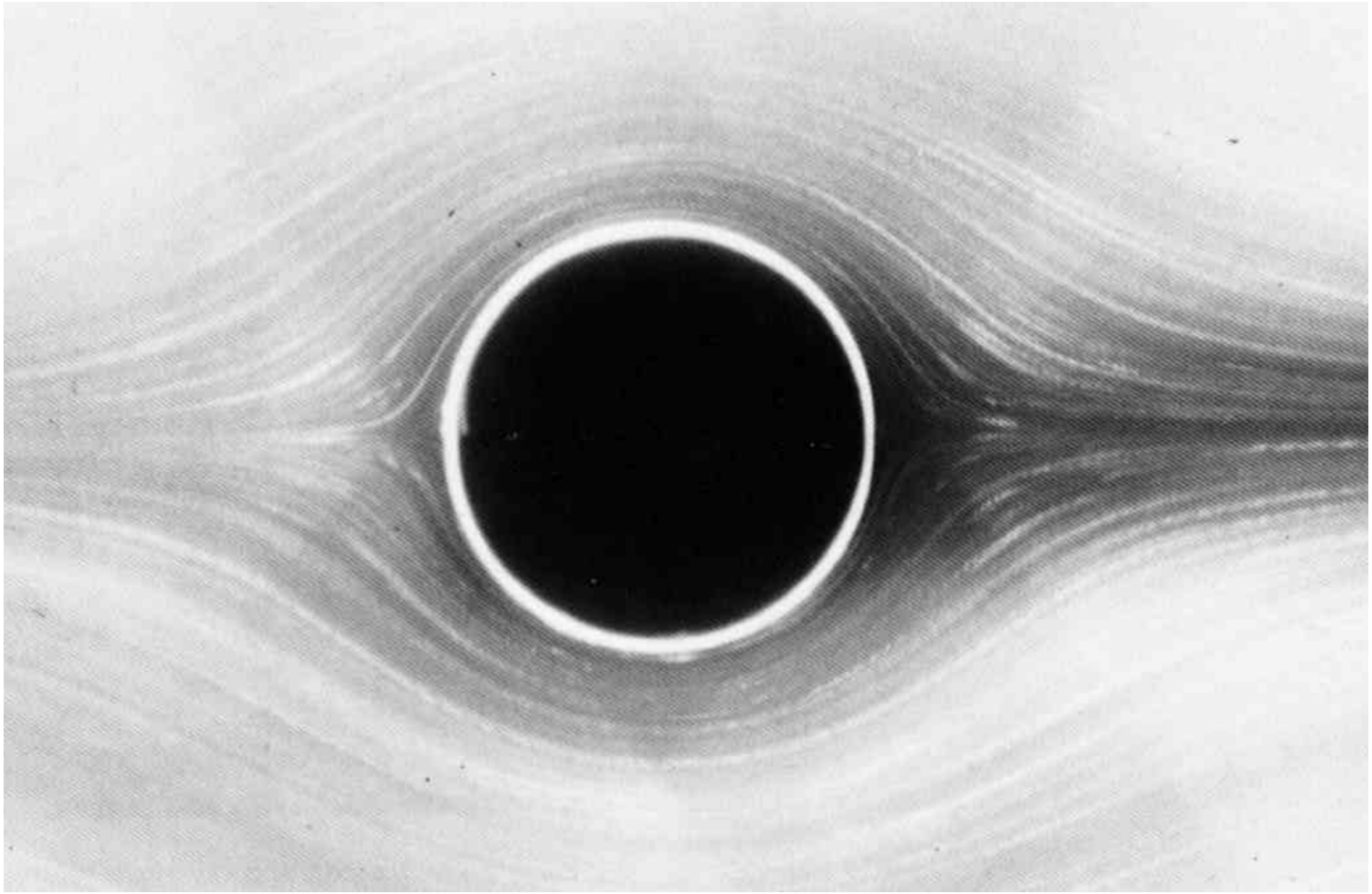
(Image credits: *An Album of Fluid Motion*, van Dyke 1982).

Onset of Turbulence



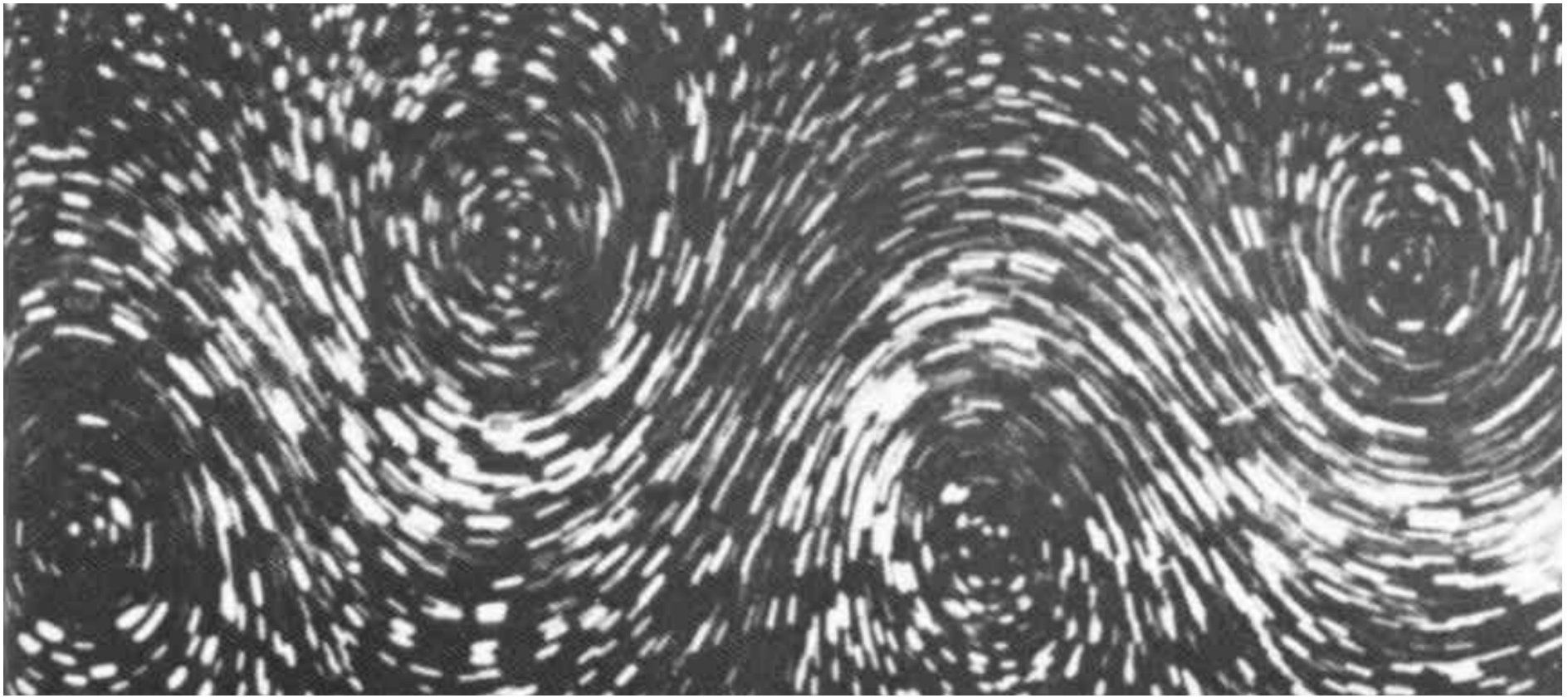
Turbulence past an obstacle at Re$= LU/\nu = 2000$ (air bubbles in water).

Laminar Flow



Shear flow past an obstacle at Re$=0.16$
(aluminum dust suspended in water).

Vortices



Vortices in von Kármán vortex street, flow past a cylinder obstacle (particles suspended in water).

Equations of Motion

Specialize Navier-Stokes equations,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p, \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

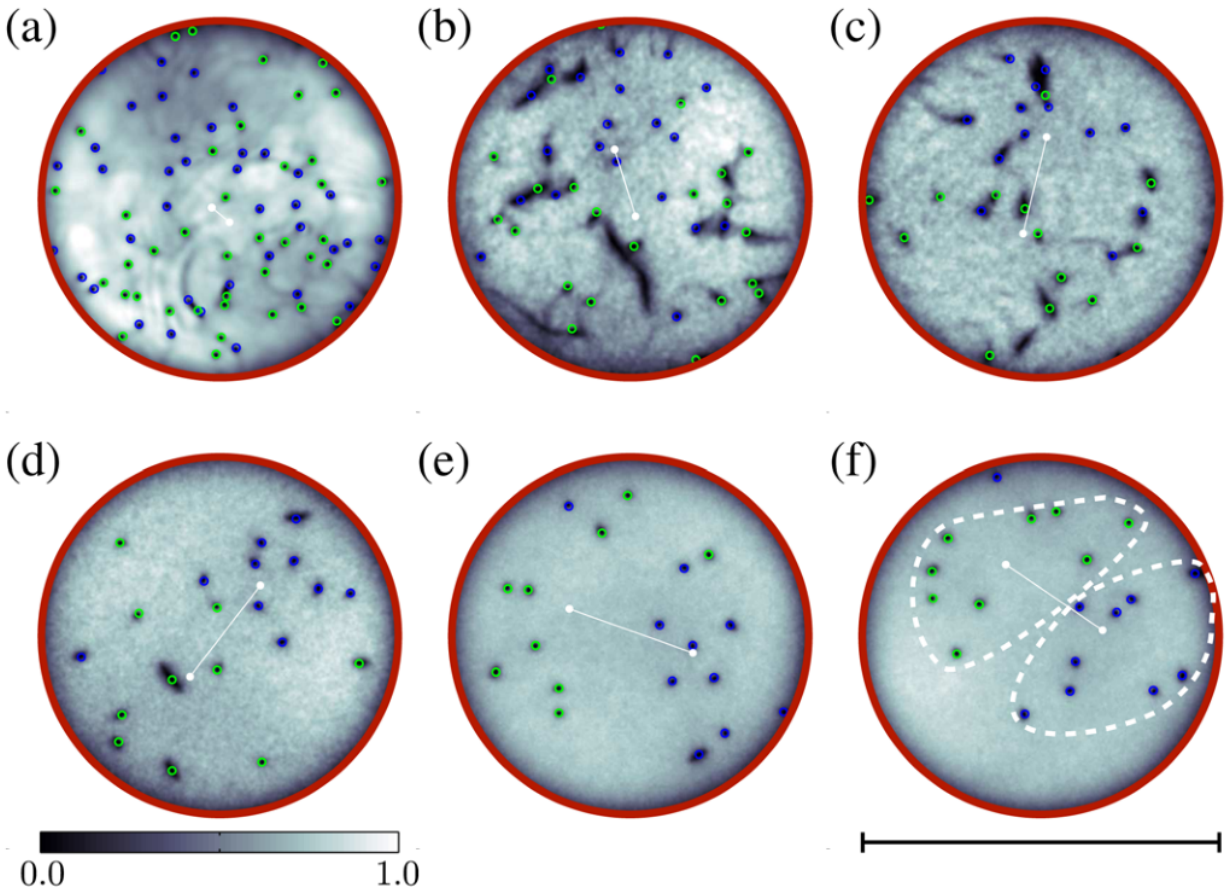
to a 2D inviscid flow (Euler equations) on a periodic domain,

$$\omega = \nabla \times u, \quad \partial_t \omega + (u \cdot \nabla) \omega = 0.$$

Vorticity distribution $\omega = \sum^N \gamma_i \delta_{x_i}$ satisfies if

$$\dot{x}_i = J \nabla_{x_i} H(x), \quad H = \sum_{i<j} \gamma_i \gamma_j G(x_i - x_j).$$

Vortex Aggregation



Wavefunction of Gross-Pitaevskii Mean-Field model of a Bose-Einstein condensate, reduced to (modified) 2D Euler equation via Madelung transform, from Simula et al. PRL 2014. Mathematically rigorous relation with PV by Lin-Xin CMP 1999.

Onsager's Negative Temperature Argument

In a finite-volume Hamiltonian system (H the Hamiltonian function) consider microcanonical volume (volume of energy (sub)levels):

$$\Phi(E) = \int_{H<E} dx, \quad Z_E = \int_{H=E} dx = \frac{d\Phi}{dE}.$$

Because of finite volume Φ' is positive and finite, it assumes maximum value at an energy level $\Phi''(E_{max}) = 0$. Inverse temperature

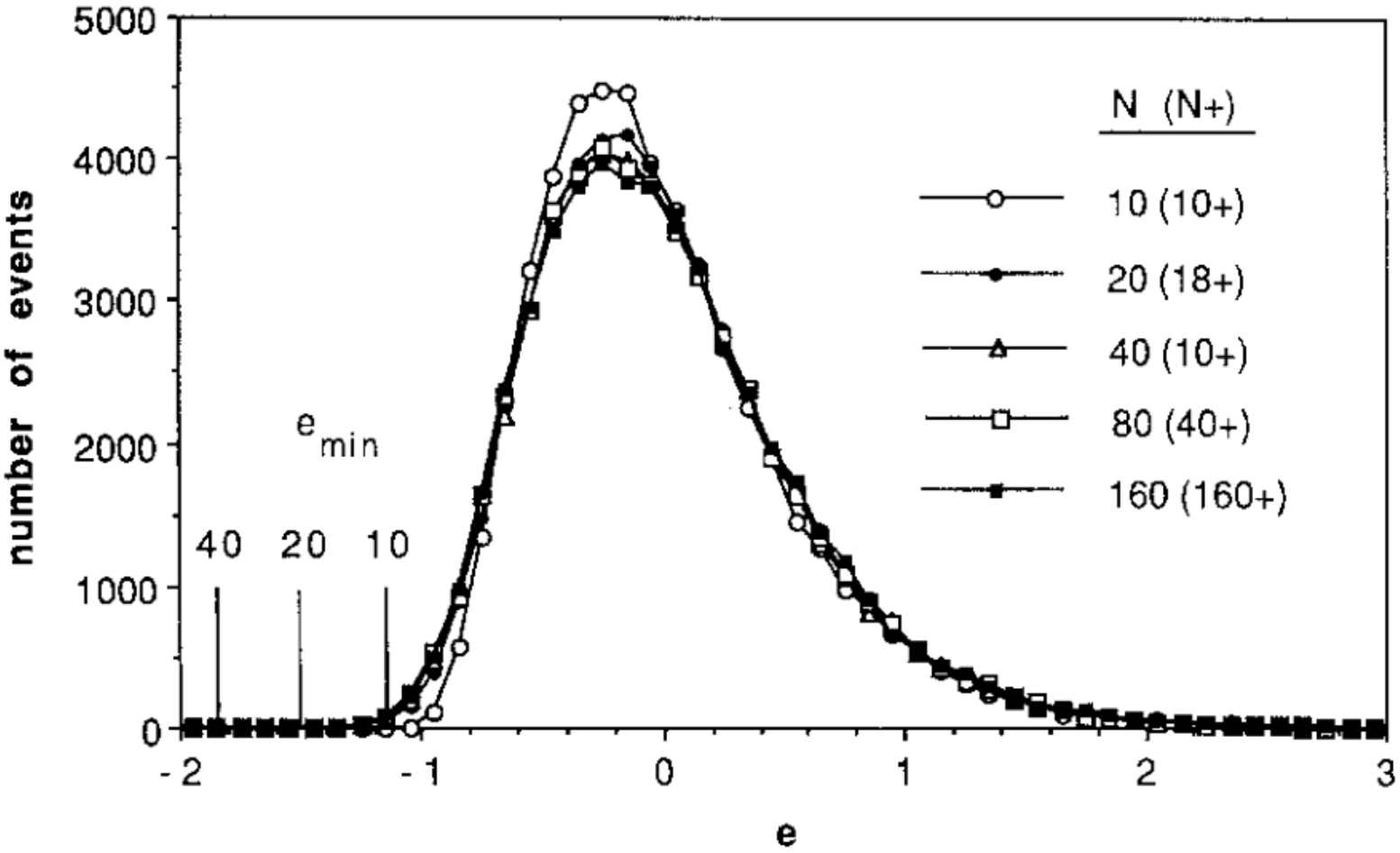
$$\frac{1}{T} = \frac{dS}{dE} = \frac{d}{dE}(k_B \log Z_E) = k_B \frac{\Phi''}{\Phi'}$$

is therefore positive for $E < E_{max}$, but *negative* for higher energies.

No negative temperature states exist in the infinite volume limit.

(cf. Fröhlich-Ruelle CMP '82, original Onsager paper in Nuovo Cimento '49).

Energy Distribution



Energy distribution of identical point vortices in a box, from Campbell & O'Neil, J. Stat. Phys. 1991.

Mean Field Limits

Consider the mean-field rescaled canonical ensemble of identical vortices

$$d\nu_{N,\beta} = \frac{1}{Z_N(\beta)} e^{-\frac{\beta}{N} H_N(x_1,\dots,x_N)} dx^N, \quad Z_N(\beta) = \int_{\mathbb{T}^N} e^{-\frac{\beta}{N} H_N} dx^N,$$

which is well-defined ($Z_N(\beta) < \infty$) for $\beta < 8\pi$. For $-8\pi < \beta < 0$,

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta)$$
$$= \sup \left\{ -\frac{\beta}{2} \int_{\mathbb{T}} \rho\,(G * \rho)\, dx - \int_{\mathbb{T}} \rho \log \rho \, dx \,\middle|\, \rho \geq 0, \int_{\mathbb{T}} \rho \, dx = 1 \right\} = 0,$$

and the maximizer is the uniform distribution. No equivalence with microcanonical ensemble (unless $E = 0$).

Absolute Equilibrium Ensembles for 2D Euler Equations

In the inviscid case ($\nu = 0$) there are two quadratic first integrals,

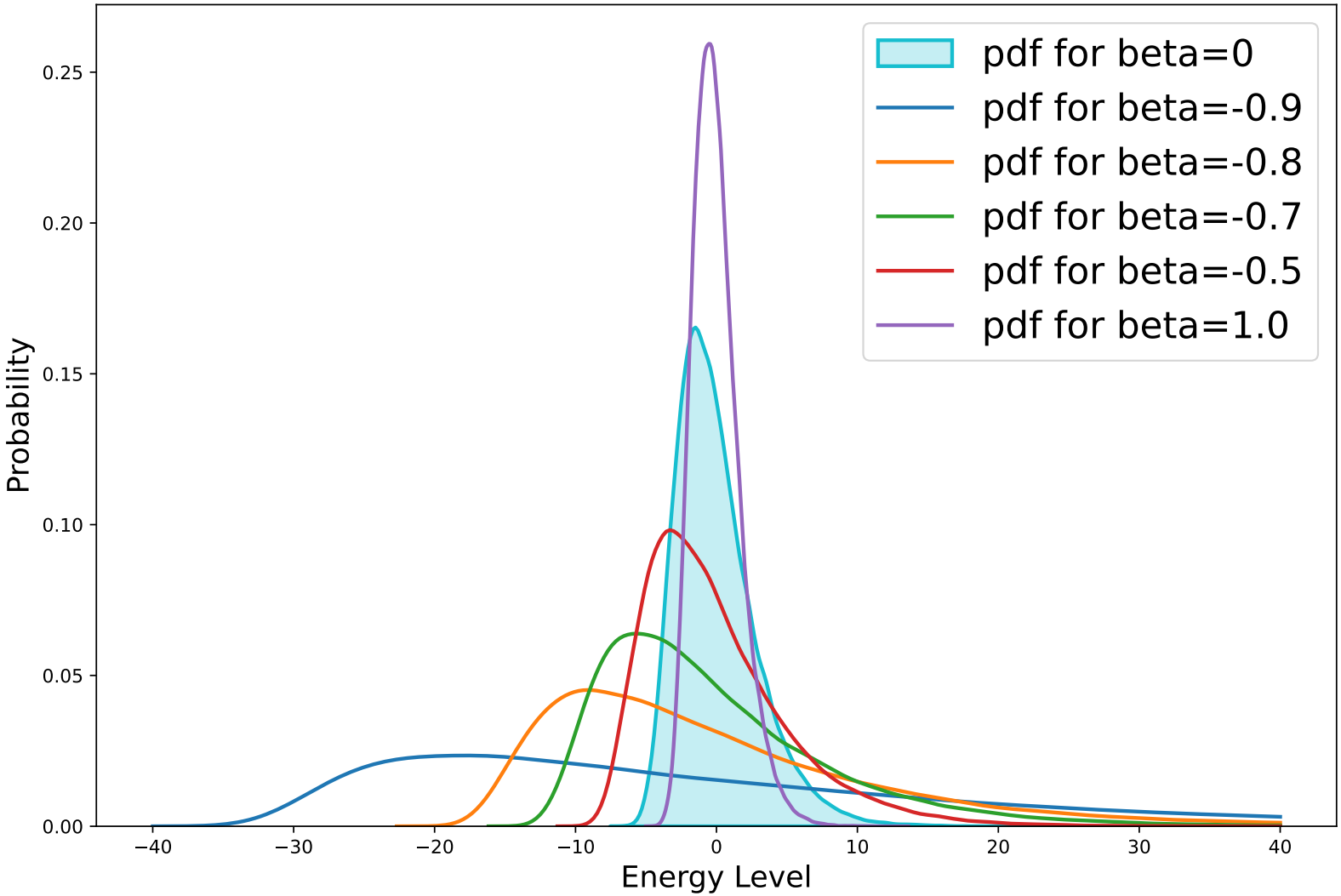
$$E = \frac{1}{2}\int |u|^2 dx = \frac{1}{2}\int \omega(-\Delta)^{-1}\omega, \qquad \mathcal{E} = \frac{1}{2}\int \omega^2 dx,$$

(vorticity is constant along flow lines of u, so actually infinitely many first integrals $I_g = \int g(\omega)dx$). For $\beta > -4\pi^2$ (spectral gap) one has energy-enstrophy equipartition (Gaussian) measures

$$d\mu_\beta(\omega) = \frac{1}{Z_\beta} e^{-\beta E(\omega) - \mathcal{E}(\omega)} d\omega,$$

$$Z_\beta = \int e^{-\beta :E:(\omega)} d\mu_0(\omega) = \det{}_2(I + \beta G)^{-1/2}.$$

Probability Density of Renormalized Energy



Gaussian ensemble with 40×40 modes, 10^5 samples kde-smoothed for each β.

Gaussian Fluctuations of Vortices

If positions are i.i.d. uniform, the vorticity distribution

$$\omega_N = \frac{1}{\sqrt{N}} \left(\sum_{j=1}^{N} \delta_{x_j} - N dx \right) \xrightarrow[N\to\infty]{\text{law on } \mathcal{D}'(\mathbb{T})} \mu_0.$$

and the Hamiltonian converges to renormalized energy,

$$H_N/N \xrightarrow[N\to\infty]{\text{law}} \tfrac{1}{2} :E(\omega): = \tfrac{1}{2} :\langle \omega, G * \omega \rangle: .$$

Then, ω_N under the canonical distribution $\frac{1}{Z_N(\beta)} e^{-\beta H_N/N} dx^N$ converges to $\mu_\beta = \frac{1}{Z_\beta} e^{-\beta :E:} d\mu_0$ if

$$\sup_{N\geq 2} Z_N(\beta) < \infty.$$

A Scaling Limit Result

Theorem

We have

$$\lim_{N\to\infty} Z_N(\beta) = Z_\beta = \det{}_2(I+\beta G)^{-1/2} = \exp\left(\frac{1}{2}\sum_{k\geq 2}\frac{(-\beta)^k}{k}\operatorname{Tr}(G^{\circ k})\right).$$

for all $\beta \geq 0$ (G.-Romito CMP '20) and a small interval $\beta_0 < \beta < 0$ (G. '26+).

The Central Limit Theorem for ω_N under the canonical ensemble is a direct consequence.

For non-negative temperature also time evolution converges, G-Luongo-Romito AAP '24

Cluster Expansion

Write the partition function as

$$Z_N(\beta) = I_N^{N(N-1)/2} W_N(\beta), \quad W_N(\beta) = \int_{\mathbb{T}^N} \prod_{1 \le i < j \le N} \big(1 + K_N(x_i - x_j)\big)\, dx^N,$$

in terms of the zero-averaged functions

$$K_N(x) = I_N^{-1} e^{\frac{\beta}{N} G(x)} - 1, \qquad I_N = \int_{\mathbb{T}} e^{\frac{\beta}{N} G(x)}\, dx,$$

The prefactor

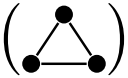
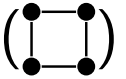
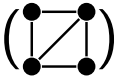
$$I_N = 1 + \frac{\beta^2}{2N^2} \int_{\mathbb{T}} G(x)^2\, dx + O(N^{-3}), \quad \binom{N}{2} \log I_N = \frac{\beta^2}{4} \operatorname{Tr}(G^{\circ 2}) + O(N^{-1}).$$

gives to leading order the $k = 2$ term in determinant expansion.

Diagram Expansion

Each factor of W_N can be represented with a diagram,

$$W_N(\beta) = 1 + A_N\left(\right) + A_N\left(\right) + A_N\left(\right) + \cdots$$

for instance

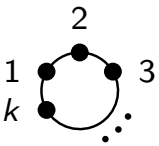
$$A_N\left(\right) = \int_{\mathbb{T}^3} K_N(x_1 - x_2) K_N(x_2 - x_3) dx_1 dx_2 dx_3 = 0.$$



Only diagrams whose connected components can not be disconnected removing an edge (1PI) have nontrivial amplitude.

Loops

The total contribution of single loops to W_N is

$$\binom{N}{k}\frac{(k-1)!}{2}A_N\left(\quad\right.$$


$$\left.\right)$$

$$= \binom{N}{k}\frac{(k-1)!}{2}\operatorname{Tr}(K_N^{\circ k}) = \frac{\beta^k}{2k}\operatorname{Tr}(G^{\circ k})(1+o(1)), \quad N\to\infty,$$

together with the prefactor this gives exactly the first term

$$Z_N(\beta) \sim \det{}_2(I+\beta G)^{-1/2} = \exp\left(\frac{1}{2}\sum_{k\geq 2}\frac{(-\beta)^k}{k}\operatorname{Tr}(G^{\circ k})\right).$$

All other diagrams can be bounded organizing them by 1-loop skeletons, but the sum is overall bounded only in a small interval $\beta_0 < \beta < 0$.

Thank you!